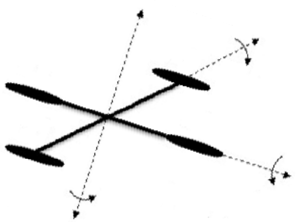
2. First-Order Differential Equations



- 변수 분리형

$$\frac{dy}{dx} = g(x)h(y)$$

$$p(y)\frac{dy}{dx} = g(x), \quad p(y) = 1/h(y)$$

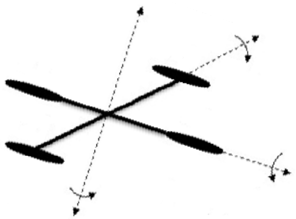
$$y = \phi(x), \quad p\left(\phi(x)\right)\phi'(x) = g(x)$$

$$\int p(\phi(x))\phi'(x)\,dx = \int g(x)\,dx.$$

$$dy = \phi'(x)dx$$

$$\int p(y)\,dy = \int g(x)\,dx \quad \text{or} \quad H(y) = G(x) + c$$

$H(y),\ G(x) \rightarrow\ p(y) = 1/h(y),\ g(x)$, antiderivatives



Example 1 Solving a Separable DE

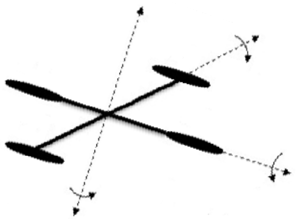
$$(1+x)\,dy - y\,dx = 0$$

Solution

Dividing; $dy/y = dx/(1+x)$

$$\int \frac{dy}{y} = \int \frac{dx}{1+x} \rightarrow \ln|y| = \ln|1+x| + c_1$$ ← laws of exponents

$$y = e^{\ln|1+x|+c_1} = e^{\ln|x+1|} \cdot e^{c_1} = |1+x|\,e^{c_1} = \pm e^{c_1}(1+x) = c_1(1+x)$$



Example 2 Solution Curve

Solve the initial-value problem $\dfrac{dy}{dx} = -\dfrac{x}{y}, \quad y(4) = -3$

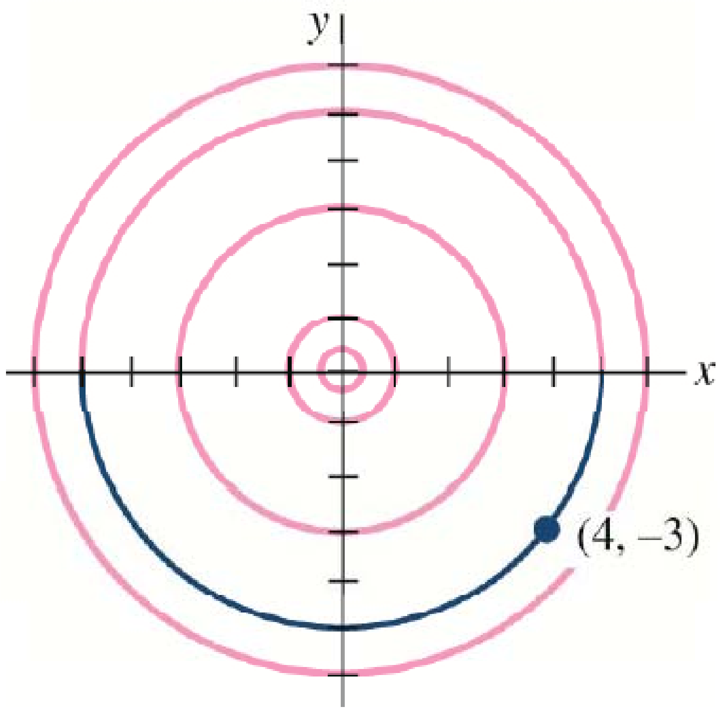
Solution

$$\int y\,dy = -\int x\,dx \quad \text{and} \quad \frac{y^2}{2} = -\frac{x^2}{2} + c_1$$

→ $x^2 + y^2 = c^2$

$y(4) = -3$ → $c = 5$

$$y = \phi_2(x), \text{ or } y = -\sqrt{25 - x^2}, \quad -5 < x < 5$$



Example 4 An Initial-Value Problem

$$\cos x(e^{2y} - y)\frac{dy}{dx} = e^{y}\sin 2x, \quad \text{solve} \;\; y(0) = 0$$

Solution

$$\frac{e^{2y} - y}{e^{y}}dy = \frac{\sin 2x}{\cos x}dx.$$

$$\int (e^{y} - ye^{-y})\,dy = 2\int \sin x\,dx$$

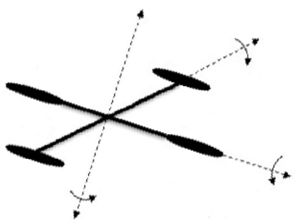
$$e^{y} + ye^{-y} + e^{-y} = -2\cos x + c.$$

$$y(0) = 0 \;\rightarrow\; c = 4$$

$$e^{y} + ye^{-y} + e^{-y} = 4 - 2\cos x.$$

- 선형 미방

$$\frac{dy}{dx} + P(x)y = f(x).$$



Example 1 Solving a Linear DE

$$\frac{dy}{dx} - 3y = 6$$

Solution

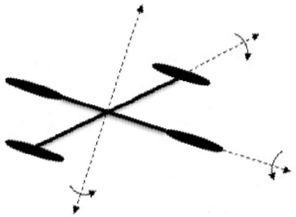
$$P(x) = -3, \quad I.F. = e^{\int P(x)dx} = e^{-3x}$$

$$e^{-3x}\frac{dy}{dx} - 3e^{-3x}y = 6e^{-3x}$$

$$\frac{d}{dx}[e^{-3x}y] = 6e^{-3x}.$$

$$e^{-3x}y = -2e^{-3x} + c$$

$$y = -2 + ce^{3x}, -\infty < x < \infty.$$



Example 2 General Solution

$$x\frac{dy}{dx} - 4y = x^6 e^x.$$

Solution

$$\frac{dy}{dx} - \frac{4}{x}y = x^5 e^x.$$

$$P(x) = -4/x, \quad I.F. = e^{-4\int dx/x} = e^{-4\ln x} = e^{\ln x^{-4}} = x^{-4} \qquad (b^{\log_b N} - N, N > 0.)$$

$$\left(x^{-4}\frac{dy}{dx} - 4x^{-5}y = xe^x. \right)$$

$$\frac{d}{dx}\left[x^{-4}y\right] = xe^x$$

$$x^{-4}y = xe^x - e^x + c \quad \text{or} \quad y = x^5e^x - x^4e^x + cx^4.$$

Example 3 General Solution

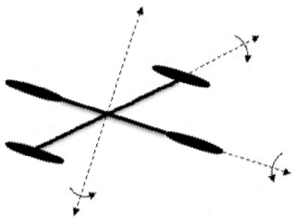
$$(x^2-9)\frac{dy}{dx}+xy=0.$$

Solution

$$P(x)=x/(x^2-9),\quad I.F.=e^{\int x\,dx/(x^2-9)}=e^{\frac{1}{2}\int 2x\,dx/(x^2-9)}=e^{\frac{1}{2}\ln|x^2-9|}=\sqrt{x^2-9}.$$

$$\frac{d}{dx}[\sqrt{x^2-9}\,y]=0$$

$$\sqrt{x^2-9}\,y=c,\quad y=c/\sqrt{x^2-9}.$$



Example 4 An Initial-Value Problem

$$\frac{dy}{dx} + y = x, \quad y(0) = 4.$$

Solution

$$P(x) = 1, \quad I.F. = e^{\int dx} = e^{x}$$

$$\frac{d}{dx}[e^{x}y] = xe^{x}$$

$$e^{x}y = xe^{x} - e^{x} + c \;\rightarrow\; y = x - 1 + ce^{-x}, \quad y(0) = 4 \;\rightarrow\; c = 5$$

$$y = x - 1 + 5e^{-x}, \qquad -\infty < x < \infty.$$

general solution

$$y = y_c + y_p = \underbrace{x-1}_{y_p} + \underbrace{ce^{-x}}_{y_c}$$

- 완전 미방

$$y\,dx + x\,dy = 0$$ → separable

$$y\,dx + x\,dy = d(xy) = 0 \;\rightarrow\; xy = c$$

Definition 2.4.1 Exact Equation

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

Theorem 2.4.1 Criterion for an Exact Differential

$M(x, y)\,dx + N(x, y)\,dy$ exact differential $\rightleftarrows$ $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}.$

Example 1 Solving an Exact DE

$$2xy\,dx + (x^2 - 1)\,dy = 0.$$

Solution

$$M(x,y) = 2xy \quad \text{and} \quad N(x,y) = x^2 - 1 \;\rightarrow\; \frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x} \;\rightarrow \text{Exact!}$$

$$\frac{\partial f}{\partial x} = 2xy \quad \text{and} \quad \frac{\partial f}{\partial y} = x^2 - 1.$$

$$f(x,y) = x^2 y + g(y).$$

$$\frac{\partial f}{\partial y} = x^2 + g'(y) = x^2 - 1.$$

$$g'(y) = -1 \quad \text{and} \quad g(y) = -y.$$

$$f(x,y) = x^2 y - y \rightarrow \; x^2 y - y = c$$

Example 2 Solving an Exact DE

$$(e^{2y} - y\cos xy)\,dx + (2xe^{2y} - x\cos xy + 2y)\,dy = 0.$$

Solution

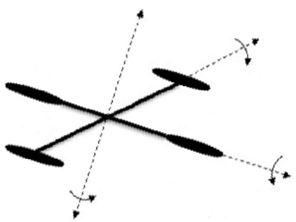
$$\frac{\partial M}{\partial y} = 2e^{2y} + xy\sin xy - \cos xy = \frac{\partial N}{\partial x}.$$ → Exact!

$$N(x,y) = \frac{\partial f}{\partial y} = 2xe^{2y} - x\cos xy + 2y$$

$$f(x,y) = xe^{2y} - \sin xy + y^2 + h(x)$$

$$\frac{\partial f}{\partial x} = e^{2y} - y\cos xy + h'(x) = M(x,y) = e^{2y} - y\cos xy,$$

→ $h'(x) = 0,\;\; h(x) = c\,,$



Last soultion: $xe^{2y} - \sin xy + y^2 + c = 0.$

Example 3 An Initial-Value Problem

$$\frac{dy}{dx} = \frac{xy^2 - \cos x \sin x}{y(1-x^2)}, \quad y(0) = 2$$

Solution

$$(\cos x \sin x - xy^2)\,dx + y(1-x^2)\,dy = 0$$

$$\frac{\partial M}{\partial y} = -2xy = \frac{\partial N}{\partial x}. \rightarrow \text{Exact!}$$

$$N = \frac{\partial f}{\partial y} = y(1-x^2), \quad f(x,y) = y^2(1-x^2)/2 + h(x)$$

$$\frac{\partial f}{\partial x} = -xy^2 + h'(x) = M = \cos x \sin x - xy^2.$$

$$h'(x) = \cos x \sin x, \quad h(x) = -\int (\cos x)(-\sin x\,dx) = -\cos^2 x/2$$

$$y^2(1-x^2)/2 - \cos^2 x/2 = c_1 \quad \text{or} \quad y^2(1-x^2) - \cos^2 x = c,$$

$$y(0) = 2 \rightarrow c = 3, \quad \boxed{y^2(1-x^2) - \cos^2 x = 3}$$